Forçability Problem

Obrad Kasum

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Contents

1 Origin of the Problem

Remark 1.1. The infinitary propositional formulas are assumed to be built from the language consisting of $\top, \bot, \neg, \implies$ and infinitary $\wedge$ and $\vee$, indexed by an arbitrary set. $\square$

Definition 1.2. We say that an infinitary formula is generically satisfiable iff a model for it can be added by forcing. $\square$

Theorem 1.3 (Gödel (essentially); Väänänen). *Let ϕ be an infinitary propositional formula. The following are equivalent:*

a. ϕ is generically satisfiable;

b. ϕ does not prove $\bot$;[1]

c. Player II has a winning strategy for the Model Existence Game associated to ϕ.[2] $\square$

Definition 1.4. We say that an infinitary formula is ssp-satisfiable iff a model for it can be added by an **ssp** forcing.[3] $\square$

Problem 1.5. *Characterize, in the spirit of Theorem 1.3, when an infinitary propositional formula is ssp-satisfiable.* $\square$

Remark 1.6. We are able to provide a game-theoretic characterization, and this is the topic of the rest of the tutorial. The problem of a proof-theoretic characterization remains open. $\square$

2 Partially Dense Filters

Definition 2.1. A (combinatorial) problem is a pair $(\mathbb{H}, \mathcal{D})$, where $\mathbb{H}$ is a poset with the property

$$(\forall w, w' \in \mathbb{H})(w \| w' \implies \inf\{w, w'\} \downarrow)$$

and $\mathcal{D}$ is a collection of pre-dense subsets of $\mathbb{H}$. $\square$

Notation 2.2. In the above scenario, if $w \| w'$, then we denote their infimum by $w \wedge w'$. $\square$

Definition 2.3. Let $(\mathbb{H}, \mathcal{D})$ be a problem.

a. We say that $(\mathbb{H}, \mathcal{D})$ is generically satisfiable iff there is a filter for $\mathbb{H}$, in some generic extension, which meets every set in $\mathcal{D}$.

b. We say that $(\mathbb{H}, \mathcal{D})$ is ssp-satisfiable iff there is a filter for $\mathbb{H}$, in some **ssp** generic extension, which meets every set in $\mathcal{D}$. $\square$

Fact 2.4. *Let $(\mathbb{H}, \mathcal{D})$ be a problem.*

a. Then $(\mathbb{H}, \mathcal{D})$ generically satisfiable.

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[1] The implied proof system is the straight forward generalization of natural deduction to infinitary propositional logic. The proof trees can be infinite, but they are well-founded.

[2] The model existence games are defined in [Vää11].

[3] The abbreviation ssp stands form "stationary-set-preserving".

b. If $\mathcal{D}$ is the set of all pre-dense subsets of $\mathbb{H}$, then $(\mathbb{H}, \mathcal{D})$ is generically satisfiable if and only if $\mathbb{H}$ is ssp. $\square$

Fact 2.5. *Let ϕ be a satisfiable infinitary propositional formula.*

- *Define $\mathbb{H}_\phi$ to consist of all finite, generically satisfiable sets of subformulas of ϕ that contain ϕ, ordered inclusion.*
- *For every sub-formula ψ of ϕ of the form $\bigwedge_{i\in I} \theta_i$ and every $i \in I$, define*

$$D_{\psi,i} = \{w \in \mathbb{H}(\phi) \,:\, \psi \in w \implies (\exists w' \leqslant w)(\theta_i \in w')\}\,.$$

- *For every sub-formula ψ of ϕ of the form $\bigvee_{i\in I} \theta_i$, define*

$$D_{\psi} = \{w \in \mathbb{H}(\phi) \,:\, \psi \in w \implies (\exists w' \leqslant w)(\exists i \in I)(\theta_i \in w')\}\,.$$

- *For every sub-formula ψ of ϕ of the form $\theta \implies \chi$, define*

$$D_{\psi} = \{w \in \mathbb{H}(\phi) \,:\, \psi, \theta \in w \implies (\exists w' \leqslant w)(\chi \in w')\}\,.$$

- *Let $\mathcal{D}_\phi$ consist of all sets of the form $D_{\psi,i}$ and D_ψ (when these are defined).*

Then $(\mathbb{H}_\phi, \mathcal{D}_\phi)$ is a problem, and this problem is ssp-satisfiable if and only if ϕ is ssp-satisfiable. $\square$

Fact 2.6. *Let $(\mathbb{H}, \mathcal{D})$ be a problem. Let $\phi_{\mathbb{H},\mathcal{D}}$ be the infinitary propositional formula, using the set*

$$\{p_w \,:\, w \in \mathbb{H}\}$$

as the set of propositional letters, which is the conjunction of the following formulas:

- $p_{1_{\mathbb{H}}}$;
- $\bigwedge_{w\in\mathbb{H}} \bigwedge_{w'\geqslant_{\mathbb{H}} w} (p_w \implies p_{w'})$;
- $\bigwedge_{w,w'\in\mathbb{H}} \bigvee_{w''\leqslant_{\mathbb{H}} w,w'} (p_w \wedge p_{w'} \implies p_{w''})$;
- $\bigwedge_{D\in\mathcal{D}} \bigvee_{w\in D} p_w$.

Then $\phi_{\mathbb{H},\mathcal{D}}$ is ssp-satisfiable if and only if $(\mathbb{H}, \mathcal{D})$ is ssp-satisfiable. $\square$

Problem 2.7. *Characterize game-theoretically when a problem $(\mathbb{H}, \mathcal{D})$ is ssp-satisfiable.* $\square$

3 Ssp-satisfiability Game

Remark 3.1. Let us fix a problem $(\mathbb{H}, \mathcal{D})$. We work towards characterizing game-theoretically when this problem is ssp-satisfiable. $\square$

Definition 3.2.

- For a set X, we denote by $\mathrm{trcl}(X)$ its transitive closure.

- A virtual model is a set M such that $M \preceq \mathrm{trcl}(M)$ and $\mathrm{trcl}(M) \models \mathrm{ZFC}^-$.
- For a virtual model M, we denote by δ_M the least ordinal which is not in M.
- For a virtual model M, we denote by λ_M the greatest λ such that $V_\lambda \in M$. $\square$

Definition 3.3. A virtual model chain is a set $\mathcal{M}$ such that:

a. $\mathcal{M}$ is finite;

b. every element of $\mathcal{M}$ is a countable virtual model;

c. for all $M \in \mathcal{M}$, it holds that λ_M is a beth-fixed point and $\mathrm{trcl}(M) = \mathrm{Hull}(M, V_{\lambda_M})$;

d. if $\delta_M = \delta_N$, then $M = N$,

e. if $\delta_M < \delta_N$, then $M \in N$ and $\lambda_M < \lambda_N$. $\square$

Definition 3.4. Let M and N be virtual models. A map $\pi : M \to N$ is called a lifting iff it is an isomorphism (w.r.t. $\in$) and there is an elementary $\pi_* : \mathrm{trcl}(M) \to \mathrm{trcl}(N)$ such that $\mathrm{crit}(\pi_*) \geqslant \lambda_M$ and $\pi_* \restriction M = \pi$. $\square$

Definition 3.5.

a. The class $\mathbb{P}^*$ consists of all $p = (w_p, \mathcal{M}_p)$ such that $w_p \in \mathbb{H}$ and $\mathcal{M}_p$ is a finite vm-chain.

b. The ordered $\leqslant$ on vm-chains is defined by asserting that $\mathcal{M} \leqslant \mathcal{N}$ holds iff for all $N \in \mathcal{N}$, there exists $M \in \mathcal{M}$ such that

   i. $\delta_M = \delta_N$,

   ii. $\lambda_M = \lambda_N$,

   iii. there exists $X \subseteq V_{\lambda_N}$ such that $M = \mathrm{Hull}(N, X)$.

c. The order $\leqslant$ on $\mathbb{P}^*$ is defined by asserting that $p \leqslant q$ holds iff $w_p \leqslant_{\mathbb{H}} w_q$ and $\mathcal{M}_p \leqslant \mathcal{M}_q$. $\square$

Definition 3.6. Suppose that $\kappa \in \beth_{\mathrm{fix}}$ and $p \in \mathbb{P}^* \cap V_\kappa$. Then the game $\boxtimes_\kappa^{\mathrm{ssp}}(p)$ is defined as the length ω two Player game of the form

$$\begin{array}{c|ccccccc} \mathrm{I} & & Q_0 & & Q_1 & & \cdots \\ \hline \mathrm{II} & p_{-1} & & p_0 & & p_1 & \cdots \end{array}$$

where $p_{-1} \in \mathbb{P}^* \cap V_\kappa$ is such that $p_{-1} \leqslant p$ and for all $n < \omega$, the following is satisfied.

a. Player I must ensure that either

   i. $Q_n = D$ for some $D \in \mathcal{D}$, or

   ii. $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary,

   iii. $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$.

b. Player II must ensure that $p_n \in \mathbb{P}^* \cap V_\kappa$ and $p_n \leqslant p_{n-1}$.

c. Player II must ensure that if $Q_n = D$ for some $D \in \mathcal{D}$, then there exists $w \in D$ such that $w_{p_n} \leqslant w$.

d. Player II must ensure that if $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary, then there exist

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{n-1}, U)$$

and $\lambda \in \beth_{\mathrm{fix}} \cap \kappa$ such that

$$\kappa \cap \mathrm{Hull}(M, V_\lambda) \subseteq \lambda,$$

$M{\downarrow}\lambda \in \mathcal{M}_{p_n}$, and $\delta_M \in S$.

e. Player I must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \longrightarrow M^*$$

such that $p_{n-1} \in \pi(E)$.

f. Player II must ensure that if $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and for some $E \in M$, then there exists $q \in E$ such that $\delta(\mathrm{Hull}(M, q)) = \delta(M)$ and $p_n \leqslant q$.

The infinite plays with no rules broken are won by Player II. ☐

Definition 3.7. Suppose that $\kappa \in \beth_{\mathrm{fix}}$. We let $\mathbb{C}_\kappa$ consist of all $p \in \mathbb{P}^* \cap V_\kappa$ such that Player II wins $\boxtimes_\kappa^{\mathrm{ssp}}(p)$. ☐

Notation 3.8. We denote by o the pair $(1_{\mathbb{H}}, \emptyset)$. ☐

Fact 3.9. *Suppose that $\kappa \in \beth_{\mathrm{fix}}$. Then the following holds.*

a. The set $\mathbb{C}_\kappa$ is an initial segment of $(\mathbb{P}^*, \leqslant, o)$.

b. If $\mathbb{C}_\kappa$ is non-empty, then $o \in \mathbb{C}_\kappa$ and $(\mathbb{C}_\kappa, \leqslant, o)$ is a poset.

c. If $\mathbb{C}_\kappa$ is non-empty, then, letting g be a V-generic for $\mathbb{C}_\kappa$, we have that the set

$$\{w_p : p \in g\}$$

is a $\mathcal{D}$-generic filter for $\mathbb{H}$.

Proof.

1° Suppose that $\mathbb{C}_\kappa \neq \emptyset$.

2° We have that for all $p \in \mathbb{C}_\kappa$, for all $w \in \mathbb{H}$ satisfying that $w \geq w_p$, it holds that $(w, \emptyset) \in \mathbb{C}_\kappa$ and $(w, \emptyset) \geq p$. This readily implies that

$$\mathbb{C}_\kappa \Vdash \text{“}\{w_p : p \in g\} \text{ is a filter in } \mathbb{H}\text{”}.$$

3° It remains to verify genericity. Let $D \in \mathcal{D}$ be arbitrary. We want to show that

$$\mathbb{C}_\kappa \Vdash (\exists p \in \dot{g})(w_p \in D).$$

4° To this end, it suffices to show that the set

$$E := \{p \in \mathbb{C}_\kappa \,:\, (\exists w \in D)(w_p \leqslant w)\}$$

is dense in $\mathbb{C}_\kappa$.

5° Let $p \in \mathbb{C}_\kappa$ be arbitrary. We want to find $q \in E$ such that $q \leqslant p$.

6° Since $p \in \mathbb{C}_\kappa$, there exists a winning strategy σ for Player II in $\boxtimes^{\mathrm{p}}_\kappa(p)$.

7° Let us consider the following partial play of $\boxtimes^{\mathrm{p}}_\kappa(p)$ according to σ.

$$\begin{array}{c|cc} \mathrm{I} & & D \\ \hline \mathrm{II} & p_{-1} & & p_0 \end{array}$$

8° Since p_{-1} and p_0 appear in a play according to a winning strategy for Player II, we have that $p_{-1}, p_0 \in \mathbb{C}_\kappa$.

9° The rules of the game imply that

$$p_0 \leqslant p_{-1} \leqslant p$$

and that there exist $w \in D$ such that $w_{p_0} \leqslant w$.

10° This imply that $p_0 \in E$, so $q := p_0$ is as required.

□

Fact 3.10. *Suppose that $\kappa \in \beth_{\mathrm{fix}}$ and that $\mathbb{C}_\kappa$ is non-empty. Then $\mathbb{C}_\kappa$ is stationary set preserving.*

Proof.

1° Let us assume otherwise. Then there exist $S \subseteq \omega_1$ which is stationary, $p \in \mathbb{C}_\kappa$, and $\dot{C}$ which is a canonical name for a subset of ω_1 such that $p \Vdash$"$\dot{C}$ is a club and $\check{S} \cap \dot{C} = \emptyset$".

2° For all $q \in \mathbb{C}_\kappa$, let σ_q be a winning strategy for Player II in the game $\boxtimes^{\mathrm{ssp}}_\kappa(p)$.

3° Consider the following partial play of $\boxtimes^{\mathrm{ssp}}_\kappa(p)$ according to σ_p.

$$\begin{array}{c|cc} \mathrm{I} & & (U, S) \\ \hline \mathrm{II} & p_{-1} & & p_0 \end{array}$$

Here, $U \subseteq H((2^\kappa)^+)$ is some canonical coding of the tuple

$$(\mathbb{H}, \mathcal{D}, (\sigma_p \,:\, p \in \mathbb{C}_\kappa), \dot{C}).$$

4° The rules then imply that $p_0 \leqslant p$ and that for some countable

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{-1}, \mathbb{H}, \mathcal{D}, (\sigma_p \,:\, p \in \mathbb{C}_\kappa), \dot{C})$$

and for some $\lambda \in \beth_{\mathrm{fix}} \cap \kappa$, we have that

$$\kappa \cap \mathrm{Hull}(M, V_\lambda) \subseteq \lambda,$$

$M{\downarrow}\lambda \in \mathcal{M}_{p_0}$, and $\delta_M \in S$.

5° **Claim.** p_0 is semigeneric for $(M, \mathbb{C}_\kappa)$.

Proof.

1′ We need to verify the following. Let $q \in \mathbb{C}_\kappa$ be such that $q \leqslant p_0$ and let E be a dense open subset of $\mathbb{C}_\kappa$ such that $q \in E \in M$. We need to find $r \in E$ such that $\delta(\mathrm{Hull}(M, r)) = \delta(M)$ and $r \| q$.

2′ Let us consider the following partial play of $\boxtimes_\kappa^{\mathrm{ssp}}(q)$ according to σ_q.

$$\begin{array}{c|ccc} \mathrm{I} & & (N, E \cap V_\lambda) & \\ \hline \mathrm{II} & q_{-1} & & q_0 \end{array}$$

Here, N is the unique virtual model in $\mathcal{M}_{q_{-1}}$ such that $\delta_N = \delta_M$. (It exists since $M{\downarrow}\lambda \in \mathcal{M}_{p_0}$ and $q_{-1} \leqslant p_0$.)

3′ Let

$$\pi : \mathrm{trcl}(M{\downarrow}\lambda) \xrightarrow{\simeq} \mathrm{Hull}(M, V_\lambda) \prec H_\theta$$

be the anti-collapse. We have that $\pi^{-1}(\kappa) = \lambda$ and $\pi^{-1}(E) = E \cap V_\lambda$.

4′ Player I did not break the rules since

a. $N \in \mathcal{M}_{q_{-1}}$,

b. $E \cap V_\lambda = \pi^{-1}(E) \in M{\downarrow}\lambda \subseteq N$,

c. $\pi{\restriction}N : N \to \mathrm{Hull}(M, X) \prec H((2^\kappa)^+)$ is a lifting, where $X \subseteq V_\lambda$ is such that $N = \mathrm{Hull}(M{\downarrow}\lambda, X)$,

d. $q_{-1} \in E = \pi(E \cap V_\lambda)$ (because E is open and $q_{-1} \leqslant q$).

5′ The rules then imply that $q_0 \leqslant q$ and that there exists $r \in E \cap V_\lambda$ such that $\delta(\mathrm{Hull}(N, r)) = \delta(N)$ and $q_0 \leqslant r$.

6′ Thus, $r \in E$, $r \| q$ (as witnessed by q_0), and $\delta(\mathrm{Hull}(M, r)) = \delta(M)$. (For the last point, note that

$$\delta(M) \leqslant \delta(\mathrm{Hull}(M, r)) = \delta(\mathrm{Hull}(M{\downarrow}\lambda, r)) \leqslant$$

$$\leqslant \delta(\mathrm{Hull}(N, r)) = \delta(N) = \delta(M).)$$

□

6° Now, let g be a V-generic filter for $\mathbb{C}_\kappa$ that contains p_0. Since p_0 is semigeneric for $(M, \mathbb{C}_\kappa)$, we have that

$$\delta_{M[g]} = \delta_M.$$

7° Since $\dot{C} \in M$ and $\dot{C}_g$ is a club in ω_1^V, we have that

$$\delta_M = \delta_{M[g]} \in \dot{C}_g.$$

8° Since $\delta_M \in S$, we get that $\dot{C}_g \cap S \neq \emptyset$, which contradicts line 1° and the fact that $p_0 \leqslant p$.

$\square$

Remark 3.11. We now turn towards showing that the ssp consistency implies the existence of a winning strategy for Player II in $\boxtimes_\kappa^{\mathrm{ssp}}(o)$, for some κ. $\square$

Lemma 3.12. *The following are equivalent.*

a. $(\mathbb{H}, \mathcal{D})$ *is ssp-satisfiable.*

b. There exists an ssp complete boolean algebra $\mathbb{B}$ *and a mapping* $i : \mathbb{H} \to \mathbb{B}$ *satisfying that*

  i. $i(1) = 1$,
  ii. for all $p, q \in \mathbb{H}$, *if* $p \leqslant q$, *then* $i(p) \leqslant i(q)$,
  iii. for all $p, q \in \mathbb{H}$, *if* $p \perp q$, *then* $i(p) \wedge i(q) = 0$,
  iv. for all $p, q \in \mathbb{H}$, *if* $p \parallel q$, *then* $i(p \wedge q) = i(p) \wedge i(q)$,
  v. for all $D \in \mathcal{D}$, $\bigvee i[D] = 1$. $\square$

Lemma 3.13. *Suppose that*

1. $\mathbb{Q}$ *is a poset,*
2. $\theta \gg \mathrm{rank}(\mathbb{Q})$ *is regular,*
3. $\mathbb{Q}$ *is stationary set preserving,*
4. $S \subseteq \omega_1$ *is stationary,*
5. U *is an arbitrary subset of* H_θ.

Then there exists a countable $M \prec (H_\theta, \in, U)$ *such that* $\delta_M \in S$ *and* $\mathbb{Q}$ *is semiproper for* M.

Proof. See [FJZ03, Lemma 4.8]. $\square$

Lemma 3.14. *Suppose that*

1. $\mathbb{Q}$ *is a poset,*
2. $\theta \gg \mathrm{rank}(\mathbb{Q})$ *is regular,*
3. $M \prec (H_\theta, \in, \mathbb{Q})$ *is countable,*
4. p *is semigeneric for* $(M, \mathbb{Q})$,
5. $p \in E \in M$.

Then there exist $r \in E \cap \mathbb{Q}$ *and* $s \leq p, r$ *such that* $s \Vdash \check{r} \in \check{M}(\dot{g})$. $\square$

Theorem 3.15. *Suppose that*

1. *there exists a proper class of inaccessible cardinals,*
2. $(\mathbb{H}, \mathcal{D})$ *is ssp-consistent.*

Then for some inaccessible κ, Player II has a winning strategy in $\boxtimes_\kappa^{\mathrm{ssp}}(o)$.

Proof.

1° There exists an ssp, complete boolean algebra $\mathbb{B}$ and a mapping $i : \mathbb{H} \to \mathbb{B}$ satisfying that

   a. for all $w \in \mathbb{H}$, $i(w) > 0$,
   b. $i(1) = 1$,
   c. for all $p, q \in \mathbb{H}$, if $p \leqslant q$, then $i(p) \leqslant i(q)$,
   d. for all $p, q \in \mathbb{H}$, if $p \perp q$, then $i(p) \wedge i(q) = 0$,
   e. for all $p, q \in \mathbb{H}$, if $p \parallel q$, then $i(p \wedge q) = i(p) \wedge i(q)$,
   f. for all $D \in \mathcal{D}$, $\bigvee i[D] = 1$.

2° Let κ be an inaccessible cardinal satisfying that $\mathrm{rank}(\mathbb{H}), \mathrm{rank}(\mathbb{B}) < \kappa$ and let us show that $o \in \mathbb{C}_\kappa$. We need to show that Player II has a winning strategy in $\boxtimes_\kappa^{\mathrm{ssp}}(o)$.

3° Let us assume otherwise. Since the game is closed for Player II, it follows that Player I has a winning strategy σ. We will reach a contradiction by showing that Player II can defeat σ.

4° For $p \in \mathbb{P}^*$ and $b \in \mathbb{B}$, let $\Psi(p, b)$ be the conjunction of the following statements:

   a. for all $M \in \mathcal{M}_p$, $\mathbb{H}, \mathbb{B}, i \in M$,
   b. for all $M \in \mathcal{M}_p$, $\lambda_M > \mathrm{rank}(\mathbb{B})$,
   c. $0 < b \leqslant i(w_p)$,
   d. for all $M \in \mathcal{M}_p$, b is semigeneric for $(M, \mathbb{B})$.

   We will show that Player II can play by maintaining that for all $n \in [-1, \omega)$, there exists $b \in \mathbb{B}$ such that $\Psi(p_n, b)$ holds.

5° Suppose first that $n = -1$. By Lemma 3.13, there exists a countable

$$M \prec (H((2^\kappa)^+), \in, \kappa, \mathbb{H}, \mathbb{B}, i)$$

   such that $\mathbb{B}$ is semiproper for M.

6° Let $\lambda := \sup(\kappa \cap M)$ and let $p_{-1} := (\emptyset, \{M{\downarrow}\lambda\})$. We see that conditions 4°a and 4°b are met.

7° Since $\mathbb{B}$ is semiproper for M and $1_{\mathbb{B}} \in M$, there exists $b \in \mathbb{B}$ such that

$$0 < b \leqslant 1 = i(w_{p_{-1}})$$

   and such that b is semigeneric for $(M, \mathbb{B})$. It is easily seen that $\Psi(p_{-1}, b)$ holds.

8° Let us now inductively consider the case n assuming that there exists $b \in \mathbb{B}$ such that $\Psi(p_{n-1}, b)$ holds. Let Player I make a move Q_n and let us show how Player II can answer in such a way as to ensure that there exists $c \in \mathbb{B}$ such that $\Psi(p_n, c)$ holds.

9° **Case I.** $Q_n = D$ for some $D \in \mathcal{D}$.

Proof.

1′ Since $\bigvee i[D] = 1$, there exists $w \in D$ such that $i(w) \wedge b > 0$.

2′ Let $p_n := (w_{p_{n-1}} \wedge w, \mathcal{M}_{p_{n-1}})$ and let $c := i(w) \wedge b$. We have that

$$0 < c \leqslant i(w_{p_n})$$

and $\mathcal{M}_{p_n} = \mathcal{M}_{p_{n-1}}$.

3′ Since also $c \leqslant b$, it follows easily that $\Psi(p_n, c)$ holds.

□

10° **Case II.** $Q_n = (U, S)$ for some $U \subseteq H((2^\kappa)^+)$ and for some $S \subseteq \omega_1$ which is stationary.

Proof.

1′ By Lemma 3.13, there exists

$$M \prec (H((2^\kappa)^+), \in, \kappa, p_{n-1}, U)$$

such that $\delta_M \in S$ and $\mathbb{B}$ is semiproper for M.

2′ Let $F : \kappa \to \kappa$ be defined by setting

$$F(\xi) := \sup(\kappa \cap \operatorname{Hull}(M, V_\xi)) < \kappa$$

for $\xi < \kappa$. There exists $\lambda \in \beth_{\text{fix}} \cap \kappa$ such that $F[\lambda] \subseteq \lambda$.

3′ Let $p_n := (w_{p_{n-1}}, \mathcal{M}_{p_{n-1}} \cup \{M{\downarrow}\lambda\}) \in \mathbb{P}^*$. We have that $p_n \leqslant p_{n-1}$ and $M{\downarrow}\lambda \in \mathcal{M}_{p_n}$. It remains to find $c \in \mathbb{B}$ such that $\Psi(p_n, c)$.

4′ By elementarity, there exists $b_M \in \mathbb{B} \cap M$ such that $\Psi(p_{n-1}, b_M)$ holds.

5′ Since $\mathbb{B}$ is semiproper for M, there exists $c \in \mathbb{B}$ such that $0 < c \leqslant b_M$ and such that c is semigeneric for $(M, \mathbb{B})$. It follows that $\Psi(p_n, c)$ holds, as required.

□

11° **Case III.** $Q_n = (M, E)$ for some $M \in \mathcal{M}_{p_{n-1}}$ and some $E \in M$.

Proof.

1′ We have that there exist $M^* \prec (H((2^\kappa)^+), \in, \kappa)$ and a lifting

$$\pi : M \to M^*$$

such that $p_{n-1} \in \pi(E)$.

2′ We need to find $p_n \leqslant p_{n-1}$, $q \in E$, and $c \in \mathbb{B}$ such that

a. $\delta(\operatorname{Hull}(M, q)) = \delta(M)$,

  b. $p_n \leqslant q$,
  c. $\Psi(p_n, c)$ holds.

$3'$ Let $\mathcal{N} := \mathcal{M}_{p_{n-1}} \cap V_{\lambda_M} \in M$ and let F consist of all $d \in \mathbb{B}$ such that there exists $q \in E \cap \mathbb{P}^*$ satisfying that

  a. for all $N \in \mathcal{M}_q$, $\mathbb{H}, \mathbb{B}, i \in N$,
  b. for all $N \in \mathcal{M}_q$, $\lambda_N > \mathrm{rank}(\mathbb{B})$,
  c. $0 < d \leqslant i(w_q)$,
  d. for all $N \in \mathcal{M}_q$, d is semigeneric for $(N, \mathbb{B})$,
  e. $\mathcal{N}$ is an initial segment of $(\mathcal{M}_q, \in)$.

  We have that $F \in M$.

$4'$ **Claim.** There exists $d \in F$ and $c \in \mathbb{B}$ such that

  a. for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, it holds that $\delta(\mathrm{Hull}(N, d)) = \delta(N)$,
  b. $0 < c \leqslant b \wedge d$,
  c. for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, c is semigeneric for $(\mathrm{Hull}(N, d), \mathbb{B})$.

  Proof.

  $1''$ Since $p_{n-1} \in \pi(E)$ and $\Psi(p_{n-1}, b)$ holds, it follows that $b \in \pi(F)$.

  $2''$ Since $F \subseteq \mathbb{B}$, we have that $F \in M \cap V_{\lambda_M}$. This means that $\pi(F) = F$ and consequently, $b \in F$.

  $3''$ By Lemma 3.14, there exists $d \in F$ and $c \in \mathbb{B}$ such that

$$0 < c \leqslant b \wedge d$$

  and $c \Vdash d \in M(\dot{g})$.

  $4''$ Let $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$ be arbitrary. We have to verify that $\delta(\mathrm{Hull}(N, d)) = \delta(N)$ and c is semigeneric for $(\mathrm{Hull}(N, d), \mathbb{B})$.

  $5''$ Since $c \leqslant b$ and b is semigeneric for N, we have that

$$c \Vdash \delta(N(\dot{g})) = \delta(N).$$

  $6''$ Since $c \Vdash d \in M(\dot{g})$, we have that $c \Vdash d \in N(\dot{g})$

  $7''$ Consequently, $c \Vdash N \prec \mathrm{Hull}(N, d) \prec N(\dot{g})$.

  $8''$ Lines $5''$ and $7''$ imply that $\delta(\mathrm{Hull}(N, d)) = \delta(N)$.

  $9''$ Line $7''$ also implies that $c \Vdash \mathrm{Hull}(N, d)(\dot{g}) = N(\dot{g})$.

  $10''$ By referencing line $5''$ once again, we get that c is semigeneric for $(\mathrm{Hull}(N, d), \mathbb{B})$.

$\square$

$5'$ Since $d \in F \cap \mathrm{Hull}(M, d)$, there exists $q \in E \cap \mathbb{P}^* \cap \mathrm{Hull}(M, d)$ such that

  a. for all $N \in \mathcal{M}_q$, $\mathbb{H}, \mathbb{B}, i \in N$,
  b. for all $N \in \mathcal{M}_q$, $\lambda_N > \mathrm{rank}(\mathbb{B})$,
  c. $0 < d \leqslant i(w_q)$,
  d. for all $N \in \mathcal{M}_q$, d is semigeneric for $(N, \mathbb{B})$,

   e. $\mathcal{M}_{p_{n-1}} \cap V_{\lambda_M}$ is an initial segment of $(\mathcal{M}_q, \in)$.

6′ Let

   a. $w_{p_n} := w_{p_{n-1}} \wedge w_q$,
   b. $\mathcal{M}_{p_n} := \mathcal{M}_q \cup \{\mathrm{Hull}(N, d) : N \in \mathcal{M}_{p_{n-1}}, \delta_N \geqslant \delta_M\}$,
   c. $p_n := (w_{p_n}, \mathcal{M}_{p_n})$.

7′ **Claim.** p_n, q, and c are as required in 2′

Proof.

1″ Since $0 < c \leqslant bd \leqslant i(w_{p_{n-1}})i(w_q)$, it follows that $w_{p_{n-1}} \| w_q$. This means that w_{p_n} is a well defined element of $\mathbb{H}$.

2″ Since $\mathcal{M}_q \in \mathrm{Hull}(M, d)$, we can easily verify that $\mathcal{M}_{p_n}$ is a vm-chain.

3″ This means that $p_n \in \mathbb{P}^*$.

4″ The fact that $p_n \leqslant p_{n-1}$ is self-evident, modulo the observation that for all $N \in \mathcal{M}_{p_{n-1}} - \mathcal{N}$, we have that

$$\delta(\mathrm{Hull}(N, d)) = \delta(N) \text{ and } \lambda(\mathrm{Hull}(N, d)) = \lambda(N).$$

5″ q was picked so that $q \in E$, while the fact that $q \in \mathrm{Hull}(M, d)$ easily implies that

$$\delta(\mathrm{Hull}(M, q)) = \delta(M).$$

6″ The fact that $p_n \leqslant q$ is also self-evident.

7″ It remains to verify that $\Psi(p_n, c)$ holds, i.e. that

   a. for all $N \in \mathcal{M}_{p_n}$, $\mathbb{H}, \mathbb{B}, i \in N$,
   b. for all $N \in \mathcal{M}_{p_n}$, $\lambda_N > \mathrm{rank}(\mathbb{B})$,
   c. $0 < c \leqslant i(w_{p_n})$,
   d. for all $N \in \mathcal{M}_{p_n}$, c is semigeneric for $(N, \mathbb{B})$.

   Parts a and b are immediate, while for c, we have

$$c \leqslant b \wedge d \leqslant i(w_{p_{n-1}}) \wedge i(w_q) = i(w_{p_n}).$$

8″ Let us explain part d. By the choice of q, we have that d is semigeneric for $(N, \mathbb{B})$ for all $N \in \mathcal{M}_q$.

9″ By the choice of c, we know that it is semigeneric for

$$(\mathrm{Hull}(N, d), \mathbb{B}),$$

   for all $N \in \mathcal{M}_q - \mathcal{N}$.

10″ Since $c \leqslant d$, the last two line imply that c is generic for $(N, \mathbb{B})$ for all $N \in \mathcal{M}_{p_n}$.

□

8′ The claim concludes verification of Case III.

□

12° The three cases show together that Player II can survive ω moves against σ. This contradicts the fact that σ is winning for Player I.

□

Corollary 3.16. *Suppose that there is a proper class of inaccessible cardinals. Then the following are equivalent.*

a. $(\mathbb{H}, \mathcal{D})$ *is ssp-consistent.*

b. For some inaccessible κ*, Player II has a winning strategy in* $\boxtimes_{\kappa}^{\mathrm{ssp}}(o)$. □

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