The Measurability of the Successor of a Rank Berkeley Cardinal

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Abstract

We present the proof that if λ is rank Berkeley and DC_λ holds, then λ^+ is measurable.

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1 Context

Global Assumption 1.1. We work in ZF. All other hypotheses are explicitly stated. □

Terminology 1.2. The word "cardinal" refers to **infinite** ordinals that are not in bijection with any lesser ordinal. □

2 The Non-stationary Ideal

Notation 2.1. Let θ be a cardinal. Then NS_θ denotes the collection of all non-stationary subsets of θ. □

Fact 2.2. *Let θ be a cardinal. Then NS_θ is an ideal on θ if and only if $\mathrm{cof}(\theta) > \omega$.* □

Fact 2.3. *Assume:*

1. *θ is a cardinal satisfying $\mathrm{cof}(\theta) > \omega$;*
2. *$\mathrm{AC}_{<\mathrm{cof}(\theta)}$ holds.*

Then NS_θ is $<\mathrm{cof}(\theta)$-complete. □

Definition 2.4. Let θ be a cardinal and let $\mu < \theta$ be a regular cardinal. A subset C of θ is a μ-club iff it is unbounded and for every increasing μ-sequence α of elements of C it holds that

$$\sup_{i<\mu} \alpha_i \in C.$$

□

Notation 2.5. Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. We denote by NS^μ_θ the collection of all subsets of θ that are disjoint from a μ-club. □

Fact 2.6. *Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. Then NS^μ_θ is an ideal on θ.* □

Fact 2.7. *Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. Suppose $\eta < \mathrm{cof}(\theta)$ is a cardinal such that AC_η holds. Then NS^μ_θ is $\leqslant\eta$-complete.* □

Notation 2.8. Let I be an ideal on a cardinal θ and let S be an I-positive subset of θ. Then $I{\restriction}S$ denotes the collection

$$\{A \in \mathscr{P}(\theta) : A \cap S \in I\}.$$

□

Fact 2.9. *Let I be an ideal on a cardinal θ and let S be an I-positive subset of θ. Then $I{\restriction}S$ is an ideal on θ.* □

Definition 2.10. Let θ be a cardinal and let $\mu < \theta$ be a regular cardinal. Then S^μ_θ denotes the set of all $\xi < \theta$ satisfying $\mathrm{cof}(\xi) = \mu$. □

Fact 2.11. *Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. Then S^μ_θ is a stationary set.* □

Lemma 2.12. *Let θ be a cardinal, let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal, and let C' be a μ-club in θ. Then there is a club C in θ such that $C \cap S^\mu_\theta \subseteq C'$.*

Proof.

1° Let C be the set

$$\{\sup(C' \cap [0, \xi]) : \xi < \theta\}.$$

2° It is easy to see that C is a club. We want to show that $C \cap S^\mu_\theta \subseteq C'$.

3° Let $\xi \in C \cap S^\mu_\theta$ be given.

4° Then there is an increasing sequence of elements of $C' \cap [0,\xi]$ whose supremum is ξ.

5° Since $\mathrm{cof}(\xi) = \mu$, this sequence contains a cofinal subsequence of length at most μ.

6° By padding, we get an increasing sequence of elements of $C' \cap [0,\xi]$ of length exactly μ whose supremum is ξ. Therefore, $\xi \in C'$.

□

Lemma 2.13. *Let θ be a cardinal, let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal, and let C be a club in θ. Then the set $C \cap S^\mu_\theta$ is a μ-club in θ.*

Proof.

1° The unboundedness follows from Fact 2.11. Let us verify the closure.

2° Let α be an increasing μ-sequence of elements of $C \cap S^\mu_\theta$ and let γ be its supremum. Then $\gamma \in C$.

3° If α is eventually constant, then the conclusion $\gamma \in C \cap S^\mu_\theta$ is immediate. Therefore, let us assume that α is not eventually constant.

4° Since μ is regular, there is a subsequence β of α of length μ which is **strictly** increasing.

5° We now have $\gamma = \sup_{i<\mu} \beta_i$, which shows that $\mathrm{cof}(\gamma) = \mu$.

6° Thus, $\gamma \in C \cap S^\mu_\theta$.

□

Fact 2.14. *Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. Then $\mathrm{NS}^\mu_\theta = \mathrm{NS}_\theta{\restriction}S^\mu_\theta$.*

Proof of direct inclusion.

1° Let $A \in \mathrm{NS}^\mu_\theta$ be given.

2° Then there is a μ-club C' such that $A \cap C' = \emptyset$.

3° By Lemma 2.12, there is a club C in θ such that

$$C \cap S^\mu_\theta \subseteq C'.$$

4° We get

$$(C')^{\complement} \cap S^\mu_\theta \subseteq C^{\complement},$$

whence

$$A \cap S^\mu_\theta \subseteq (C')^{\complement} \cap S^\mu_\theta \subseteq C^{\complement} \cap S^\mu_\theta \in \mathrm{NS}_\theta{\restriction}S^\mu_\theta.$$

□

Proof of reverse inclusion.

1° Let $A \in \mathrm{NS}_\theta \restriction S^\mu_\theta$.

2° This means $A \cap S^\mu_\theta \in \mathrm{NS}_\theta$, so there is a club C such that

$$A \cap S^\mu_\theta \cap C = \emptyset.$$

3° By Lemma 2.13, we have that $C \cap S^\mu_\theta$ is a μ-club, so A is disjoint from a μ-club. In other words, $A \in \mathrm{NS}^\mu_\theta$.

□

Definition 2.15. Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. A subset of θ is μ-stationary iff it intersects every μ-club in θ. □

Fact 2.16. *Let θ be a cardinal and let $\mu < \mathrm{cof}(\theta)$ be a regular cardinal. Then every μ-stationary subset of θ is stationary.* □

3 Saturation

Definition 3.1. Suppose:

1. θ is a cardinal;
2. I is an ideal on θ;
3. μ is another cardinal.

The ideal I is μ-saturated iff there is **no** anti-chain of cardinality μ in the Boolean algebra $\mathscr{P}(\theta)/I$. □

Fact 3.2. *Suppose:*

1. *θ and μ are cardinals;*
2. *AC_μ holds;*
3. *I is a $<\mu$-complete ideal on θ.*

*Then I is μ-saturated if and only if there is **no** μ-sequence of pairwise disjoint I-positive subsets of θ.*

Proof of direct implication.

1° Assume towards contradiction that there is a μ-sequence of pairwise disjoint I-positive sets. Let S be one such sequence.

2° Then $\{[S_i]_I : i < \mu\}$ is an anti-chain of cardinality μ in $\mathscr{P}(\theta)/I$ – contradiction.

□

Proof of reverse implication.

1° Assume towards contradiction that I is not μ-saturated. Let A be an anti-chain of cardinality μ in $\mathscr{P}(\theta)/I$.

2° Let S be an A-sequence satisfying

$$(\forall a \in A)(S_a \in a)$$

(which exists by AC_μ).

3° Then for all $a \in A$, the set S_a is I-positive and for all $b \in A - \{a\}$ it holds that $S_a \cap S_b \in I$.

4° Let $e : \mu \longrightarrow A$ be an injection and for $i < \mu$ let

$$T_i = S_{e_i} - \bigcup_{j<i} S_{e_j}.$$

5° Then T is a μ-sequence of pairwise disjoint I-positive sets – contradiction.

□

4 Stationary Atoms

Definition 4.1. Let θ be a cardinal and let $A \subseteq \theta$. Then A is an atom in θ iff A is stationary and A is not a disjoint union of two stationary sets. □

Fact 4.2. *Suppose:*

1. *θ is a cardinal such that* $\mathrm{cof}(\theta) > \omega$*;*
2. *S is a subset of θ.*

Then S is an atom in θ if and only if S is stationary and $\mathrm{NS}_\theta{\restriction}S$ *is a maximal ideal.* □

5 Embeddings

Definition 5.1. Let M and N be transitive sets. Then $\mathcal{E}(M, N)$ consists of all elementary embeddings $j : M \longrightarrow N$ such that for some $\xi < o(M)$ it holds that $j(\xi) > \xi$. □

Definition 5.2. Let M and N be transitive sets and let $j \in \mathcal{E}(M, N)$. The critical point of j, denoted by $\mathrm{crit}(j)$, is the least ordinal $\xi < o(M)$ with the property that $j(\xi) > \xi$. □

Notation 5.3. Let α be an ordinal and let $j \in \mathcal{E}(V_\alpha, V_\alpha)$. Then $\lambda(j)$ denotes the ordinal

$$\sup_{n<\omega} j^n(\mathrm{crit}(j)).$$

□

Fact 5.4. *Let α be an ordinal, let $j \in \mathcal{E}(V_\alpha, V_\alpha)$, and let $\xi < \alpha$. Then:*

a. if $j(\xi) = \xi$, then $\xi < \mathrm{crit}(j)$ or $\xi \geqslant \lambda(j)$;

b. if $\xi \in \mathrm{crit}(j) \cup \{\lambda(j)\}$, then $j(\xi) = \xi$. □

6 Main

Fact 6.1. *Suppose:*

1. $\mu \leqslant \lambda$ *are cardinals;*
2. AC_μ *holds.*

Then $\mathrm{cof}(\lambda^+) > \mu$.

Proof.

1° Assume towards contradiction that there is a cofinal map

$$f : \mu \longrightarrow \lambda^+ - \{0\}.$$

2° There is a μ-sequence g such that for all $i < \mu$ it holds that g_i is a surjection from λ onto $f(i)$.

3° Let Γ be the canonical bijection of λ onto $\mu \times \lambda$.

4° Then the function

$$\lambda \longrightarrow \lambda^+ : \xi \mapsto g_{\Gamma(\xi)_0}(\Gamma(\xi)_1)$$

is a surjection – contradiction.

□

Notation 6.2. Let α be an ordinal and let $j \in \mathcal{E}(V_\alpha, V_\alpha)$. We denote by $\mathrm{Fix}(j)$ the set

$$\{\xi < \alpha : j(\xi) = \xi\}.$$

□

Lemma 6.3. *Suppose:*

1. α *is an ordinal;*
2. $j \in \mathcal{E}(V_\alpha, V_\alpha)$*;*
3. $\theta \in \mathrm{Fix}(j)$ *is a cardinal;*
4. $\mu < \min\{\mathrm{crit}(j), \mathrm{cof}(\theta)\}$ *is a regular cardinal.*

Then $\mathrm{Fix}(j) \cap \theta$ *is a* μ*-club in* θ. □

Fact 6.4. *Suppose:*

1. λ *is a cardinal;*
2. $j \in \mathcal{E}(V_{\lambda^+ + \omega}, V_{\lambda^+ + \omega})$ *is such that* $\mathrm{crit}(j) < \lambda$*;*
3. $\mathrm{AC}_{\mathrm{crit}(j)}$ *holds.*

Then for all regular $\mu < \mathrm{crit}(j)$ *the ideal* $\mathrm{NS}^\mu_{\lambda^+}$ *is* $\mathrm{crit}(j)$*-saturated.*

Proof.

1° Assume towards contradiction that there is a $\mathrm{crit}(j)$-sequence of pairwise disjoint μ-stationary sets. Let S be one such sequence.

2° Up to replacing S_0 by

$$S_0 \cup \left(\lambda^+ - \bigcup_{i<\mathrm{crit}(j)} S_i\right),$$

we may assume that

$$\lambda^+ = \bigcup_{i<\mathrm{crit}(j)} S_i.$$

3° It follows that the set $j(S)_{\mathrm{crit}(j)}$ is μ-stationary.

4° By Lemma 6.3 and Fact 2.14, we get $j(S)_{\mathrm{crit}(j)} \cap \mathrm{Fix}(j) \neq \emptyset$. Let ξ be an element of this set.

5° Therefore: $\xi \in j(S)_{\mathrm{crit}(j)}$.

6° On the other hand, there is $i < \mathrm{crit}(j)$ such that $\xi \in S_i$.

7° We now have

$$\xi = j(\xi) \in j(S_i) = j(S)_i,$$

whence $j(S)_i \cap j(S)_{\mathrm{crit}(j)} \neq \emptyset$ – contradiction.

□

Theorem 6.5. *Suppose:*

1. *λ is a rank Berkeley cardinal;*
2. *DC_λ holds;*
3. *$\mu < \lambda$ is a regular cardinal.*

Then $S^\mu_{\lambda^+}$ contains an atom in λ^+.

Proof.

1° Assume otherwise.

2° Let $j \in \mathcal{E}(V_{\lambda^+ + \omega}, V_{\lambda^+ + \omega})$ be such that $\mathrm{crit}(j) \in (\mu, \lambda)$ and let $\kappa = \mathrm{crit}(j)$.

3° Let $[\mathcal{T}]$ consist of all trees T satisfying the following:

- elements of T are non-empty subsets of λ^+;
- the order on T is the reverse inclusion;
- $S^\mu_{\lambda^+}$ is the root of T;
- if a node in T is non-stationary, then it is a leaf of T;

- if A is a stationary element of T whose rank is **not** the greatest rank in T, then it has exactly two children, which form a stationary partition of it;
- if η is a limit ordinal strictly less than the height of T and if S is an η-sequence such that:
  - for all $i < \eta$, the set S_i is an element of T of rank i;
  - $\{S_i : i < \eta\}$ is a chain in T;
  - $\bigcap_{i<\eta} S_i \neq \emptyset$;

  then $\bigcap_{i<\eta} S_i$ belongs to T.

4° The order $\leqslant_{\mathcal{T}}$ is given by the sub-tree relation.

5° We let $\mathcal{T} = (\lfloor \mathcal{T} \rfloor, \leqslant_{\mathcal{T}})$. It is easily seen that $\mathcal{T}$ is a non-empty tree.

6° **Claim.** Let $T \in \mathcal{T}$. Then there is **no** κ-sequence S such that:

- for all $i < \kappa$ it holds that S_i is an element of rank i in T;
- $\{S_i : i < \kappa\}$ is a chain in T.

Proof.

1′ Assume otherwise, and let S be such a sequence.

2′ For all $i < \kappa$, let A_i be the child of S_i which is not on the branch S.

3′ Then $(A_i : i < \kappa)$ is a disjoint κ-sequence of μ-stationary subsets of λ^+, so $\mathrm{NS}^{\mu}_{\lambda^+}$ is **not** κ-saturated.

4′ This contradicts Fact 6.4.

□

7° Claim 6° and DC_λ yield that $\mathcal{T}$ has a maximal element. Let us fix one such element, call it T.

8° Since T is maximal in $\mathcal{T}$, every leaf in it is either an atom or a non-stationary set.

9° Claim 6° and DC_λ ensure that T is well-orderable and $|T| \leqslant \kappa$.

10° Let L be the set of leaves of T.

11° **Claim.** $S^{\mu}_{\lambda^+} = \bigcup_{A\in L} A$.

Proof.

1′ Assume otherwise. Let $\alpha \in S^{\mu}_{\lambda^+} - \bigcup_{A\in L} A$.

2′ Notice that for all $i < \kappa$, there is a unique $A_i \in T$ of rank i such that $\alpha \in A_i$.

3′ Then the sequence $(A_i : i < \kappa)$ contradicts Claim 6°.

□

12° From 9°, Claim 11°, and the fact that NS_{λ^+} is $\leqslant\kappa$-complete, we get that there is $A \in L$ which is stationary. Thus, A is an atom.

□

Corollary 6.6. *Suppose:*

1. *λ is a rank Berkeley cardinal;*
2. *DC_λ holds.*

Then λ^+ is measurable.

Proof.

1° By Theorem 6.5, there is an atom $S \subseteq S^\omega_{\lambda^+}$.

2° By Fact 4.2, the ideal $\mathrm{NS}_{\lambda^+}{\restriction}S$ is maximal.

3° It is easy to see that $\mathrm{NS}_{\lambda^+}{\restriction}S$ is not principal.

4° By Fact 6.1, λ^+ is regular.

5° It follows from Fact 2.3 that $\mathrm{NS}_{\lambda^+}{\restriction}S$ is $<\lambda^+$-complete.

6° Therefore, the dual filter of $\mathrm{NS}_{\lambda^+}{\restriction}S$ witnesses that λ^+ is measurable.

□

Credits

The proof of Corollary 6.6 comes from [Gol24], which attributes it to Woodin.

References

[Gol24] Gabriel Goldberg. "Measurable cardinals and choiceless axioms." English. In: *Ann. Pure Appl. Logic* 175.1 (2024). Id/No 103323, p. 26. ISSN: 0168-0072. DOI: 10.1016/j.apal.2023.103323. arXiv: 2106.05916 [math.LO].

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